Ends and Coends

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In these notes I will introduce the notions of ends and coends following the definitions given in [1] and [2]. These are objects which can be thought of as generalisations of limits and colimits respectively. An end can be thought of as a universal wedge so I will start by defining a wedge.

1 Wedges

Definition 1.1. Let C and D be categories with functor $T(-,-)\colon C^{\mathrm{op}} \times C \to D$. A **wedge** of T consists of an object $e \in \mathrm{ob}(D)$ and a collection of morphisms $\{\omega_a\colon e \to T(a,a)\}_{a \in \mathrm{ob}(C)}$ such that for every morphism $f\colon a \to b$ in C the following diagram commutes.

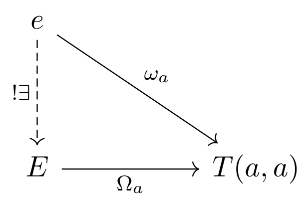
$$\begin{array}{ccc} e & \xrightarrow{\omega_a} & T(a,a) \\ {\scriptstyle \omega_b}\downarrow & & \downarrow{\scriptstyle T(\mathrm{id}_a, f)} \\ T(b,b) & \xrightarrow[T(f,\mathrm{id}_b)]{} & T(a,b) \end{array}$$

Example 1.2. Let T be the hom functor $C(-,-)\colon C^{\mathrm{op}} \times C \to \mathbf{Set}$. For every morphism $f\colon a \to b$, a wedge (e,ω) of T is required by definition to satisfy $\omega_a \circ C(\mathrm{id}_a, f) = C(f, \mathrm{id}_b) \circ \omega_b$. Re-written, that is $\omega_a \circ f = f \circ \omega_b$. Note that this is exactly the condition for a collection of morphisms ω to comprise a natural transformation from id_C to itself. Hence, we can think of e as some set of natural transformations $\mathrm{id}_C \to \mathrm{id}_C$.

2 Ends

Now we know what a wedge is we can provide the definition of an end.

Definition 2.1. Let $T(-,-)\colon C^{\mathrm{op}} \times C \to D$ be a functor of categories C and D. The **end** of T is a wedge (E,Ω) which satisfies the universal property that for any wedge (e,ω) and object $a \in \mathrm{ob}(C)$, there exists a unique map such that following diagram commutes.



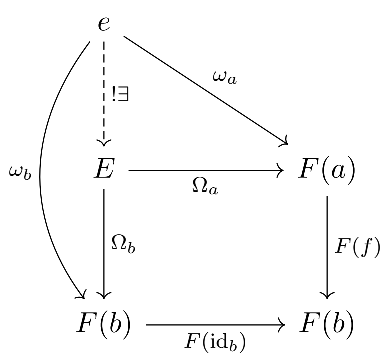
This universal property provides uniqueness of E up to isomorphism. The object of the end is denoted $\int_{c \in C} T(c, c)$.

Now we will take at look at some examples.

Example 2.2. 1. Consider the end of the hom functor $C(-,-)\colon C^{\mathrm{op}} \times C \to \mathbf{Set}$. By Example 1.2 we know that the wedges of $C(-,-)$ correspond to sets of natural transformations. The universal property ensures that $\int_{c \in C} C(c, c) \cong \mathrm{NT}(\mathrm{id}_C, \mathrm{id}_C)$ because if there were natural transformations missing there would be wedges that wouldn't factor through and if any natural transformation appears twice then a set containing just one element corresponding to that natural transformation would give at least two maps into the end rather than a single unique one.

2. Let F and G be functors $C \to D$ then consider the functor $T = D(F^{\mathrm{op}}(-), G(-))\colon C^{\mathrm{op}} \times C \to \mathbf{Set}$ where F^{op} is the functor $C^{\mathrm{op}} \to D^{\mathrm{op}}$ induced by F. For reasons similar to the first example, the end $\int_{c \in C} T(c, c)$ is isomorphic to the set of natural transformations $\mathrm{NT}(F, G)$.

3. Let T be a functor which does nothing in its first component and sends its second component to the image under a functor $F\colon J \to C$, i.e. $T(a, b) = F(b)$. To see what is happening here let us substitute this functor into the diagram which defines the end E. For any a, b and morphism $f\colon a \to b$ in the category J the following diagram commutes.



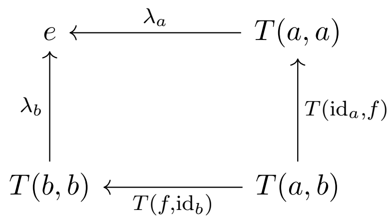
Since this diagram commutes, when we identify the objects $F(b)$ we recover the definition of a limit E of some diagram F over an indexing category J. It is in this sense which ends generalise limits.

4. An end may not necessarily exist for a given functor. For example the functor F from the empty category to the category of fields does not admit a limit because there are no terminal objects in the category of fields. The functor F corresponds to a functor $T(c, c') = F(c')$ which does not admit an end or we would have a limit of F and hence a terminal object of fields.

3 Cowedges and Coends

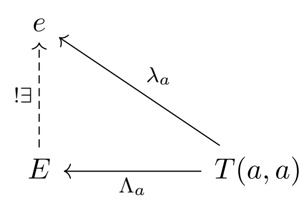
As with a many definitions in category theory, ends have a dual notion [1].

Definition 3.1. Let C and D be categories with functor $T(-,-)\colon C^{\mathrm{op}} \times C \to D$. A **cowedge** of T consists of an object $e \in \mathrm{ob}(D)$ and a collection of morphisms $\{\lambda_a\colon T(a,a) \to e\}_{a \in \mathrm{ob}(C)}$ such that for every morphism $f\colon b \to a$ in C the following diagram commutes.



Much like with ends, a coend is a universal cowedge.

Definition 3.2. Let $T(-,-)\colon C^{\mathrm{op}} \times C \to D$ be a functor of categories C and D. The **coend** of T is a cowedge (E, Λ) which satisfies the universal property that for any cowedge (e, ω) and object $a \in \mathrm{ob}(C)$ the following diagram commutes.



The object of the coend is denoted $\int^{c \in C} T(c,c)$.

Example 3.3. 1. Similarly to how we can write limits as ends in Example 2.2, we can take a colimit of a functor $F\colon C \to D$ by defining $T(a,b) = F(b)$ and taking a coend. In exactly the same way as in the case of limits and ends, we have that the colimit of F is the coend $\int^{c \in c} T(c,c)$.

2. Let's consider the coend of $\mathbf{Set}(-,-)\colon \mathbf{Set}^{\mathrm{op}} \times \mathbf{Set} \to \mathbf{Set}$. Since the coend $E \coloneqq \int^{c\in\mathbf{Set}} \mathbf{Set}(c,c)$ contains the image of every morphism λ_a we can consider E to be the disjoint union $\bigsqcup_{c\in C} \mathbf{Set}(c,c)$ modulo some equivalence relation $\sim$. Our cowedge diagram exactly says that for every morphism $f\colon b \to a$ we need $f \circ \lambda_a = \lambda_b \circ f$ and so we need $(\alpha\colon a \to a) \sim (\beta\colon b \to b)$ whenever there exists a morphism $f\colon b \to a$ such that $f\alpha = \beta f$. The universal property for a coend tells us E must contain all elements of $\bigsqcup_{c\in C} \mathbf{Set}(c,c)/\sim$.

3. For a group G, consider the one object category BG where the morphisms are group elements composed by multiplication. The coend of $BG(a,b) : BG^{\mathrm{op}} \times BG \to \mathbf{Set}$ as above is the set $\bigsqcup_{c\in C} BG(c,c)/\sim$ where $(\alpha\colon a \to a) \sim (\beta\colon b \to b)$ if and only if there exists $f : b \to a$ such that $f\alpha = \beta f$. For this category however, there is only one hom set and $f\alpha = \beta f$ implies $\alpha = f^{-1}\beta f$ which is exactly to say that $\alpha \sim \beta$ if they share a conjugacy class in G. Hence, the coend $\int^{c\in BG} BG(c,c)$ is the set of conjugacy classes of G.

References

[1] Saunders Mac Lane. "Special Limits". In: *Categories for the Working Mathematician*. Springer New York, 1978, pp. 222–229. ISBN: 978-1-4757-4721-8. DOI: 10.1007/978-1-4757-4721-8_10.

[2] Simon Willerton. *The n-Category Café*. 2014. URL: https://golem.ph.utexas.edu/category/2014/01/ends.html (visited on 06/28/2023).